A discussion on learning arithmetic geometry

with advice from Matthew Emerton, Pete Clark, and Emmanuel Kowalski

taken from https://terrytao.wordpress.com/career-advice/learn-and-relearn-your-field/

Anonymous:

Dear Prof. Tau,

First of all, I should thank you for your nice and useful hints and advices. Then, I would like to tell you a serious problem Im facing with and ask you to help, if you dont mind: Im a PhD student and my field is arithmetic algebraic geometry. As you know, the literature in this area is vast. So if I want to learn my field and go through all the details of all results and proofs, then I guess, I can never (i.e., in a reasonable time) work on my own thesis problem and produce anything. But, not learning and reading that way give me the feeling that Im missing something and Im not confident anymore.

I would really appreciate if you could give me some advice. Thank you in advance.

Matthew Emerton:

Dear Anonymous,

I hope you wont mind someone else providing you with some advice.

Most workers in arithmetic algebraic geometry (and not just students) suffer from the problem you describe to various degrees. The literature is indeed vast, and to read everything as a student, even everything that you might need in solving your particular thesis problem, is essentially impossible.

I would suggest the following: a good grounding in algebraic geometry is essential. Most students in algebraic geometry, of all flavours, go through the rite of passage known as "Hartshorne": reading Hartshornes book, especially chapters 2 and 3, and solving vast numbers of exercises. It is more or less impossible, and in any case probably unwise, to avoid doing this. And once you have solved many/most of the Hartshorne problems, you should have some baseline confidence in algebraic geometry, scheme theory, and cohomology.

At the same time, there are other texts that it is good to look at because they emphasize certain functorial aspects of algebraic geometry more than Hartshorne, aspects which are particularly important in arithmetic algebraic geometry e.g. Mumfords red book. It is advisable to supplement your Hartshorne reading with such books.

Another standard text to read is Cornell-Silverman (and these days, depending on your precise direction of interest, Cornell-Silverman-Stevens but this is more number-theoretic, while Cornell-Silverman is more geometric). This is not such a long book, and has a lot of information in it. Furthermore, since it is devoted to exposing Faltings proof of the Mordell and Tate conjectures, you get to see how all the geometric machinery is applied to solving a particular problem. As Ill comment on more in a moment, this is crucial. (And I should also add, that there is no need to read this entire book for example, below I will advocate skipping the chapter on Neron models, unless you really dont want to.)

One thing that I would advise *not* doing, for most students, is reading large amounts of EGA and SGA. This takes a lot of time, and there is real danger of not getting anywhere substantive. In particular, it is safe, at least at the beginning of your career, to learn etale cohomology (say) as a black box. (Later, if it turns out that you need detailed information about how it is constructed, you can go back and learn them.)

What *is* worthwhile, is to get a good understanding of sheaf cohomology in the classical setting. (The beginning of Borels book on intersection homology, which ultimately is about perverse sheaves and so on, but which begins with background on constructible sheaves and Grothendiecks six operations, is one place to do this.) The point is that *most* applications of etale cohomology use just the same sheaf theoretic formalism as one has in the classical setting (i.e. varieties over the complexes, with their complex topology), and the main technical theorems in the subject (proper base-change, smooth acyclicity, nearby and vanishing cycles) are precisely intended to show that etale cohomology, etale constructible sheaves, and Grothendiecks six operations in the etale setting, behave exactly as they do in the classical setting. So if one has a good understanding of sheaves in the classical setting, you can be confident that your intuition there will carry over to the etale setting.

So one thing that is very much worth studying is Delignes first paper on the Weil conjecures. There you will see how he uses etale cohomology to prove a terrific theorem, and you will see that most of what he uses are properties that have perfect analogues (and are not so hard to establish) in the classical setting. So a good intuition for classical sheaf theory will let you understand a lot of the proof.

I could summarize this aspect of my advice as follows: spend time learning things that have a wide range of application (and thus take some advantage of economies of scale): basic homological algebra and sheaf theory is one of these things it underlies coherent sheaf cohomology (as in Hartshorne), etale cohomology and sheaf theory, perverse sheaves, D-modules, , all of which are tools in arithmetic algebraic geometry. On the other hand, dont spend lots of time learning technical details in a narrow direction until you are sure you will need them.

For the next aspect, I want to return to a point I made above: one way to learn an area is, rather than learning its technical details and foundations, is to learn how it can be applied to help solve problems. For example, p-adic Hodge theory is another tool which plays a big role in a lot of arithmetic algebriac geometry, and which has a technically formidable underpinning. But, just like etale cohomology, it has a very nice formalism which one can learn to use comfortably without having to know all the foundations and proofs.

Neron models of abelian varieties are similar: one almost never needs to use any facts about their construction (other than that they exist) when applying them. So it is safe to treat their existence as a black-box. (And if it turns out that you really need the proof for something, there is an article about it in Cornell-Silverman.) What is important is to understand how their existence can be used as a tool in other arguments. Because of my own mathematical background, the most natural place for me to point to is the literature on modular curves and modular forms due to Mazur, Ribet, and Wiles. In particular, the first couple of sections of Ribets famous Inventiones 100 article give a great example of how hundreds of pages of theory (including a lot of the theory of Neron models, and a quite a bit of SGA 7) can be summarized in ten or so pages of “working knowledge”.

If you ask other people, they will be able to give similar references for other topics that you might need, which summarize “everything you need to know” in a short number of pages, rather than the hundreds of pages of original sources.

Finally, what do you do to build your confidence, given that youre skipping all these hundreds of pages?

For this, its good to remember that being a research mathematician is in any case not ulimately about reading and learning mathematics (although that plays a role), but about doing mathematics. So in some sense your confidence as a research mathematician can (at least in principal) be somewhat orthogonal to the amount of foundational proofs youve assimilated.

What you need, rather (as Ive already said above), is to understand how some important techniques can be applied to solve interesting problems.

One way to do this is by starting as soon as possible to look at the research literature.

Your advisor can suggest papers, and (depending on your precise interests) you can also choose “classics” of your own: Delignes first Weil conjectures paper, Ribets Inventiones 100 paper, Faltings paper in Cornell-Stevens, Serres paper in Duke 54 about his conjecures on modular forms and Galois representations, or any number of others. Try to find papers whose topic is appealing to you, which seem well-written, and which you feel you might have some shot of understanding something about (at least the statement of the main theorem) but dont expect to understand much of the technical heart of the paper at the beginning. Your goal is to get a feeling for how it is possible to marshal all the forces of the abstract theory to solve actual problems, by seeing someone else do it. It will take a lot of time and patience, and careful study, to do this but in the end it should pay off.

One thing to pay attention to is the bibliography it may lead you to other sources which explain necessary background material. Pursuing the necessary background by begining at the top and then working down, rather than trying to build everything from the ground up, is generally more efficient. (It is typically what working mathematicians do when they need to learn something new begin with a paper of interest, and then go back into the foundational literature just enough to fill in those points they couldnt understand from reading the paper itself.)

Another crucial way to build confidence (much more effective than learning new things!) is to solve problems yourself. You can begin with Hartshornes exercises, and any other exerices you can find scattered around. But at some point you will need more specialized problems to work on. You can ask your advisor to give you questions to solve. (And some advisors work this way in any case: rather than beginning all out with a thesis-level problem, they begin by having their students solve smaller, more manageable problems.)

But also, once you are looking at the research literature, you have an essentially endless supply of problems: just take any paper you are looking at, find a technical lemma whose hypotheses you can understand, and see if you can prove the lemma yourself. Try not to "cheat" by reading the given proof (but you may want to glance at what follows, just to check that the so-called lemma isnt actually a five page argument). But, if you cant do it yourself after a serious effort, you will be in a much better position to understand and appreciate the authors argument and whatever trick or technique they use will be one that you probably will always remember in the future!

Doing this kind of exercise is one way that working mathematicians develop the skill of being able to glance over a paper in their field and then know essentially all the details of the paper. (A skill which I found extremely impressive when I was a student!)

And of course you can try to create and solve problems of your own. (Another skill which is important to develop.) Since Ive already gone on much too long, I wont say more about this here.

I hope this advice is of some use.

Regards,

Matthew

Pete L. Clark:

Matt Emertons advice is excellent. The only thing I might add is that Qing Lius recent book "Algebraic Geometry and Arithmetic Curves" looks like a legitimate alternative to Hartshorne for arithmetically minded students of algebraic geometry. However, it is true that the treatment of sheaves and cohomology in Hartshornes book is especially comprehensive (compared to Lius book and, for that matter, any other introductory text on the subject I know of, including also Mumfords very fine Red Book), so I would recommend to a student to read selected parts of Chapters II and III of Hartshorne no matter what.

The references you asked for:

MR1047143 (91g:11066) Ribet, K. A. On modular representations of Gal($\overline{Q}/Q$) arising from modular forms. Invent. Math. 100 (1990), no. 2, 431476. (Reviewer: Glenn Stevens) 11G18 (11F32 11F80 11S37)

MR0885783 (88g:11022) Serre, Jean-Pierre(F-CDF) Sur les reprsentations modulaires de degré 2 de Gal($\overline{Q}/Q$). (French) [On modular representations of degree 2 of Gal($\overline{Q}, Q$)] Duke Math. J. 54 (1987), no. 1, 179230. 11F11 (11G05 14G15 14G25 14K15)

Matthew Emerton

Just to add one more suggestion for arithmetic geometry exercises:

Silvermans two books on elliptic curves have a lot of terrific excercies, covering a wide range of number theory and geometry. (While the geometry is naturally focused to a large extent on curves, the second book has a chapter on elliptic surfaces, for example.) These exercises vary quite a bit in difficulty, and are a good place to start for problems of a more arithmetic nature than those in Hartshorne.

And for reading: good survey articles can often give a huge increase in efficiency when learning a new field. Even if the survey doesnt give all the details, it will guide through the more technical foundational literature, and save you flailing around with hundreds of pages trying to sort out whats what.

The standard places to find surveys are: ICM proceedings, and more generally, other conference proceedings. In particular, the AMSs "Proceedings of Symposia" series has many great conference proceedings chock full of interesting survey articles. In particular, every so often there is a big conference on algebraic geometry in the U.S., and the proceedings appear in that series. Just writing from memory, there is Arcata from '74, Bowdoin from '85, and the Seattle motives conference from some time in the '90s. These have lots of surveys on all kinds of algebraic geometry, including a lot of arithmetic geometry. There is also the great volume from the early 70s on the Hilbert problems (which includes Katzs survey of Delignes proof of the Riemann hypothesis over finite fields, and one of Langlands first papers on Shimura varieties).

Reading surveys cant substitute for reading more thorough treatments of a topic, but as well as giving a guide to these more thorough treatments, it also provides a way to learn quickly about different areas of mathematics, and so build up your general knowledge of mathematics. So I would recommend that as a grad student, it is good to supplement your detailed technical reading with well-written surveys on a range of interesting topics.

Regards,

Matthew

Anonymous:

Pete L. Clark: Thank you very much for your comments and for listing the two papers.

A little comment regarding Hartshornes book: its a good book, but when I studied it for the first time, I struggled very badly with it. It was mostly my fault, but I think the book lacks crucial motivations in some parts. For example, the section on proper maps was especially tough for me; it never even mentions that properness is the analogue for compactness in topology. That may be obvious for someone who has studied complex manifolds or Riemann surfaces, but to a student who has not studied those, its hard. Another thing is that its also difficult to learn from Hartshorne how to think about schemes and their associated constructions such as fiber products. The functor of points concept is very useful for understanding schemes, but the book doesnt discuss it. So in short, Ive found that Hartshorne is good for someone who has had good preparation in geometry, but its hard otherwise.

Matthew Emerton:

Dear Anonymous,

Regarding Hartshorne, I agree with you. For those trying to come to grip with algebraic geometry for the first time (and this typically means coming to grips with Hartshorne), I think there are various ways to try to make things smoother.

One is to read other texts like Mumfords Red Book in conjunction with Hartshorne. It helps a lot with motivation for the underlying ideas. (Eisenbud and Harris is probably also good for this, although I dont know it very well I think it just post-dates my time as a grad student.)

The other thing to do is to make sure you have some understanding of classical topology and geometry. For example, there is a notion of proper map in topology: the preimage of a compact set should be compact. One can check that proper maps are precisely those that are universally closed (working in some reasonable category of topological spaces let me not be to precise here, and thus avoid making a mistake!).

Similarly, one can check that a Hausdorff space is one whose diagonal map is a closed embedding.

In short, one can translate lots of familiar and simple topological and geometric notions into more categorical terms than those in which they are usually formulated. Having done this, the corresponding categorical notions in scheme theory become a lot more intuitive.

Unfortunately, I dont know where this kind of comparison is made in the literature I think it might be something that people who think about these sort of things rediscover for themselves at various points. Although it is not really a "trick", it might be the kind of thing that someone could write up for the "trick wiki", or some companion wiki on basic mathematical concepts, if such a thing is in the offing.

This is also the basis for my earlier recommendation about learning classical sheaf theory before learning the basics of etale sheaf theory. Etale sheaf theory is largely about constucting a subsitute for classical constructible sheaf theory in the world of schemes. If you dont know classical sheaf theory then you will be at a double disadvantage in trying to learn (or use) the etale theory not only do you have to deal with the technical baggage; you dont even know why youre carrying all this baggage around!

One more comment on Hartshorne: many students, especially arithmetic geometry students, tend to focus on Chapters 2 and 3 (which of course are the technical heart of the book) and ignore Chapter 4 (on curves) and especially Chapter 5 (on surfaces). But these final two chapters have lots of beautiful geometry in them, and reading them can supply quite a bit of motivation to go back and do battle with Chapters 2 and 3. And you also learn things that will help later in life. For example, understanding models of curves over rings of integers (which, as schemes, are two-dimensional) becomes much easier if you have a little feeling for the theory of algebraic surfaces!

Also, Mumfords book "Lectures on curves on an algebraic surface" is absolutely beautiful (as are all his books on geometry), and gives a lot of motivations (both intuitive and technical) for all kinds of aspects of scheme theory (including Hilbert schemes, Picard schemes, base-change theorems for coherent cohomology, the use of nilpotents and deformation theory,). I highly recommend it.

Regards,

Matthew

Anonymous:

Dear Matthew,

Thanks for more of your advice. As I said, I wish I had all of your advice when I first had the idea of learning algebraic geometry. Its kind of late, but I still find many of your comments valuable now.

Regarding your suggestion on learning classical topological and geometrical notions in category theory terms, there is a set of lectures notes on AG by David Cox which does exactly that and is wonderful to read. One can find them here: http://www3.amherst.edu/∼dacox/. Its also important to include these ideas in a Wiki page so everyone knows where to find them.

Regards and Thanks

Emmanuel Kowalski:

Its also worth noting that a lot of the algebraic and arithmetic technology is sufficiently explicitable that various software packages can do many computations both for algebraic geometry (and commutative algebrac) and arithmetic algebraic geometry (and algebraic number theory), in particular this is the case with elliptic curves and modular forms. This is a recent developpment (and it is due to the amazing work of many people), and it can be exploited by students to get more intuition about all these theories.

(One exception seems to be tale cohomology where little is apparently computable, as witnessed by the enormous amount of work required in the work of Couveignes, Edixhoven, and otthers, to compute the Ramanujan tau function at prime arguments using Delignes results that show where it can be found in étale land).

Sheaves:

Dear Prof. Emerton,

Thank you for your advice! I have read them and bookmarked this page!

I have one question, though. I am planning on doing a PhD in abstract algebraic geometry (as of now, I only have a vague idea of what abstract AG is but I tend to go with the rule:

the more abstract, the better) rather than arithmetic algebraic geometry. I am woundering if you still advice not to jump directly to EGA or SGA after having done Hartshorne.

I guess Serres paper on coherent sheaves is a good thing to read too.

Thank you.

Matthew Emerton:

Dear Sheaves,

Serres paper on coherent sheaves is certainly great. There is also a very nice survey on sheaf cohomology on algebraic geometry written by Zariski. (It appeared in the Bulletin of the AMS in the late 50s, I think, as part of the proceedings of a workshop on algebraic geometry you should be able to find it on MathSciNet. It is also in his collected works; volume 4, I think.)

Zariski himself never used sheaves or cohomology; his work belongs to an earlier era. But in his survey he carefully explains the basics of sheaf cohomology, but more importantly, he illustrates its power as a tool by discussing several examples of geometric theorems of himself and others which are easily established via cohomological arguments. (One good example that he gives explains why in Hartshorne a lemma about the vanishing of H^1 in certain contexts is called the lemma of Enriques-Severi-Zariski, when none of these three geometers ever worked with a cohomology group.)

I think that especially if you want to work in a general and abstract setting, this survey might be useful, but it will help give you a sense of what certain ideas mean geometrically, which can be hard to intuit from the more algebraic formalism of algebraic geometry as presented in Hartshorne.

Another paper you might want to read is Serres GAGA paper. In this paper one sees how a rather abstract argument based on sheaf cohomology allows one to deduce concrete results such as Chows theorem, stating that any complex analytic subvariety of $P^n(C)$ is actually an algebraic subvariety.

As to how much time to spend on EGA and SGA, this is something that you ultimately have to decide for yourself, hopefully with the guidance of your thesis adviser. But one thing to remember is that many very clever people have pored over the details of EGA and SGA for many years now, and it so it is going to be hard for anyone to find interesting new results that can be obtained just by applying the ideas from these sources alone (as important as those ideas are). Even if you want to make progress in a very general, abstract setting, you will need ideas to come from somewhere, motivated perhaps by some new phenomenon you observe in geometry, or number theory, or arithmetic, or . . ..

This is part of the reason why I advise against spending too much time just holed up with EGA and SGA. By themselves, they are not likely (for most people) to provide the inspiration for new results. On the other hand, when you are trying to prove your theorems, you might well find techincal tools in them which are very helpful, so it is useful to have some sense of what is in them and what sort of tools they provide. But you will likely have to find your inspiration elsewhere, and so for this reason alone you will probably want to make time to look at other things too.

Regards,

Matthew